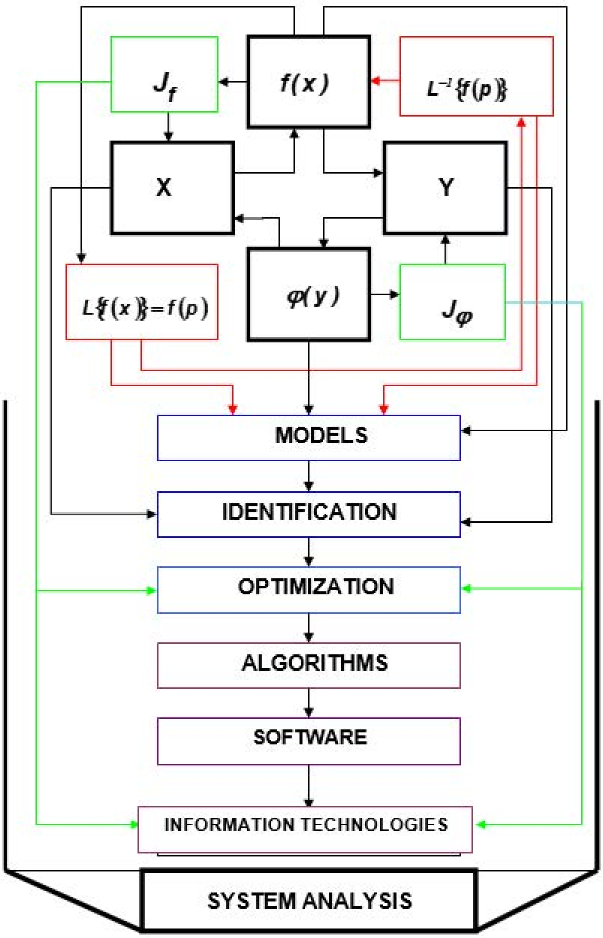
B. I. Mokin, V. B. Mokin, O. B. Mokin

FUNCTIONAL ANALYSIS IN INFORMATION TECHNOLOGIES



Ministry of Education and Science of Ukraine
Vinnytsia National Technical University

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FUNCTIONAL ANALYSIS IN INFORMATION TECHNOLOGIES

Textbook

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The textbook outlines the basics of functional analysis adapted to the solution of applied problems in the field of information technology using programs implemented in the Python language.

The textbook is recommended for English-taught students, post-graduate students specializing in the IT field in specialties 124 – “System Analysis” and 126 – “Information Systems and Technologies”.
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INTRODUCTION

Higher mathematics for non-mathematical specialties in higher education institutions contains five mathematical components with different level of detail - linear algebra, mathematical analysis, probability theory, functions of a complex variable and vector algebra. But this is not enough for the effective learning of some special subjects in the field of information technologies, and therefore the curricula of some of these specialties contain other mathematical components, among which an important role is played by the mathematical component called "Functional analysis" and which, in fact, is the "second floor" over the "Mathematical Analysis" component.

It is known from mathematical analysis that a function is a law according to which one numerical set corresponds to another numerical set.

Graphically, this can be displayed as shown in fig. B.1.

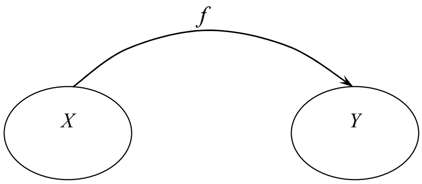

Figure B.1 – Graphical interpretation of the term function

Conventionally, the function is most often written as follows

$$y = f(x),\ x \in X,\ y \in Y, \quad (1)$$

or

$$y = y(x),\ x \in X,\ y \in Y ,, \quad (2)$$

where $\in$ –is the symbol of the element belonging to the set.

If the function f assigns only one number $y \in Y$ to each number $x \in X$, then, as is known from mathematical analysis, such a function is called a *single-valued*, and if the function assigns two or more numbers to each number, then such function is called a *multi-valued*.

A function can be specified in the form of a table, a graph, or one or more formulas.

A function the graph of which has no discontinuities belongs to the continuous class, and a continuous function the graph of which does not contain breaks and therefore has a continuous first derivative belongs to the *smooth* class.

A continuous function whose graph has breaks, and therefore its derivative - the breaks of the 1st type, belongs to the class of piecewise smooth.

From the same subject of "Mathematical analysis" it is known that a functional is a law according to which a set of functions is matched to a set of numbers.

Graphically, it looks as shown in fig. B.2.

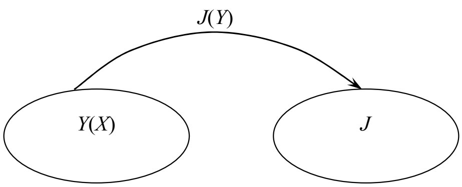

Figure B.2 – Graphical interpretation of the concept of functional

Conventionally, the functional is most often written as follows

$$J_y = J(y(x), x),\ \ x \in X,\ \ y(x) \in Y(X),\ \ J_y \in J . \tag{3}$$

Examples of functional can be definite integrals:

$$J_y = \int_a^b y(x)\, dx \tag{4}$$

or

$$J_y^f = \int_a^b f(x, y)\, dx , \tag{5}$$

or

$$J_y^F = \int_a^b F(x, y, y')\, dx , \tag{6}$$

in which $f(x, y)$ – is a mathematical expression that is a construction from an independent variable x and its function $y(x)$, and $F(x, y, y')$ – is a mathematical expression that is a construction from an independent variable x, its function $y(x)$, and the first derivative $y'(x)$ of this function; at the same time, the segment $[a, b]$ is the domain of the function $y(x)$, i.e. $x \in [a, b]$.

So the function sets the law according to which each element from one numerical set is matched with some element from another or the same numerical set, and the functional sets the law according to which each element from the set of functions is matched with some element from the set of numbers.

Then, there is a question: "Is it not possible to find a law according to which each element from a set of functions is matched by some element from another or the same set of functions?"

The answer to this question is positive, and the mathematical concept that characterizes such a law is called an *operator* in mathematics.

For example, between a set of continuous functions on a segment $[a, b]$ and a set of derivatives of these functions $f(t), t \in [a, b]$ there is a one-to-one correspondence, which is given by the differentiation operator $D = \frac{d}{dt}$, for example, the function

$$y = t^2 \tag{7}$$

corresponds to the derivative

$$\frac{dy}{dt} = 2t, \tag{8}$$

which is also a function of the same independent variable.

Analyzing the program of the educational subject "Mathematical analysis", it is easy to realise that this mathematical discipline is dedicated to the study of the properties of functions and operations with them, the main of which are differentiation and integration. And it does not pay attention at all to the study of the properties of functional and operators as independent mathematical objects. Therefore, a separate mathematical component called "Functional analysis" is dedicated to the study of these mathematical objects, which is studied by students of all mathematical specialties at universities and which is also included in the list of mandatory mathematical disciplines for students and postgraduates of some IT specialties.

The mathematical discipline “Functional analysis”, which is studied by students of mathematical specialties at universities, is a set of concepts and theorems that combine these concepts into a single mathematical structure, and therefore it contains 90 percent of the material dedicated to the formulation and proof of these theorems . At the same time, it is more important for IT students to be able to use this material in practical applications. That is why we built our study guide using material dedicated mainly to the presentation of the main concepts and final results obtained in the theory of functional analysis and their application to the solution of the applied problems that IT specialists face with. The program material of the subject is presented in six chapters, the first of which is dedicated to sets, metric spaces and their characteristics; the second – theories of measure and integrals of Riemann, Lebesgue and Stieltjes; the third – to functional and methods of finding their unconditional extrema; the fourth – methods of finding conditional extrema of functional; the fifth – theory and applied aspects of the use of operators; the sixth – characteristics and recommendations for the application of several special operators, such as direct and inverse Laplace operators and autoregressive operators, which are widely used in system analysis and applied information technologies.

In conclusion to this brief introduction to functional analysis it is necessary to note that in the English-language version of the textbook the authors used all the references listed in the bibliography, but without specification of the source, which is typical of monographs and scientific papers. And the material which is taken from the Ukrainian-language manuals, written by the authors themselves about the basics of functional analysis and some specific subjects in which the concepts of functional analysis are used, which we use in this textbook to demonstrate the solution of specific applied problems, we present without quotation marks and references.

The differences between this textbook and other study guides on functional analysis is, first of all, in a different structuring of the study material and its selection since this textbook is focused on solving those applied problems, that a specialist in information technologies confronts with. Moreover, each applied problem is accompanied by the developed computer program for implementing its algorithms in the Python language.

Chapter 1. SETS AND METRIC SPACES, THEIR CLASSES AND CHARACTERISTICS

1.1 Sets, subsets and their characteristics

The concept of a **set** in mathematics is understood as a collection of objects of a certain nature, which are called its elements. A **set** is given if all its elements and the rule according to which the elements belong to the **set** are known.

The elements of the **set** can be, for example, all the rivers flowing through Ukraine, or all natural numbers on the number line, or all real numbers located on the segment [0,1] of the number line, or all continuous functions whose arguments are given on this segment of the number axis.

In mathematics, **sets** are denoted by uppercase letters of the Latin or Greek alphabets, and their elements are denoted by lowercase letters from the same alphabets, for example, $A, B, X, Z, \mathrm{E}, \Phi, \Omega, \Psi$ – are **sets**, and $a, b, x, z, \varepsilon, \phi, \omega, \psi$ – are elements. A symbolic entry indicates $x \in X$, that an element belongs to a **set**, and a symbolic entry indicates $x \notin X$ – that it does not belong to a **set**. A **set** with a finite number of elements is called a **finite set**, and a **set** with an infinite number of elements is called an **infinite set**. An example of a **finite set** is the **set** of cars registered in Ukraine, and an example of an **infinite set** is the **set** of real numbers on the segment [0,1] of the number axis. If the elements of the **set** A are a finite numerical sequence with n members, then symbolically it can be written as $A=\{a_i\},\ i=1,\ 2,\ ...,\ n$. If the elements of this set A are an infinite numerical sequence of members, then it can be symbolically written in the form $A=\{a_i\},\ i=1,\ 2,\ 3...$. A **Set** that does not contain any element, is called an **empty set** and is denoted by the symbol $\emptyset$ or O, which does not need to be equated with the number "zero".

If the **sets** A and B consist of the same elements, then they are considered equal, as evidenced by the record $A = B$. If not all the elements of the **set** A are included in the **set** B, then the **set** A is called a **subset** of the **set** B, as evidenced by the record $A \subset B$. For example, on the number line, the **set** of rational numbers R, each of which is known to be the ratio of two integers, is a **subset** of the **set** Z of real numbers. If we are not sure that the **subset** of the **set** A contains fewer elements than the **set** B, then we write it like this: $A \subseteq B$.

When two **sets** A and B are combined, a new **set** M is formed, which is called their **sum** and which contains all the elements of both of these **sets**, and each identical element of both **sets** is included in their **sum** M as one element - symbolically, the sum is written as follows:

$$M = A \cup B \tag{1.1}$$

For example, if A and B and are numerical **sets**, where

$$A = \{1,2,3,4,5\},\quad B = \{4,5,6,7,8\}, \tag{1.2}$$

then, according to (1.1), we will have

$$M = A \cup B = \{1,2,3,4,5\} \cup \{4,5,6,7,8\} = \{1,2,3,4,5,6,7,8\} \tag{1.3}$$

At the intersection of two **sets** A and B, a new **set** P is formed, which is called their intersection and which contains only those elements of both **sets** that are the same, and each of these identical elements of both **sets** is included in their intersection as one element - the intersection is symbolically written as follows:

$$P = A \cap B \tag{1.4}$$

For numerical **sets** (1.2) given in the conditions of the previous example, according to (1.4), we have

$$P = A \cap B = \{1,2,3,4,5\} \cap \{4,5,6,7,8\} = \{4,5\} \tag{1.5}$$

For the sum and the intersection of **sets** A, B, C the following properties are valid:

Asociality

$$(A \cup B) \cup C = A \cup (B \cup C), \tag{1.6}$$

$$(A \cap B) \cap C = A \cap (B \cap C), \tag{1.7}$$

Commutativity

$$A \cup B = B \cup A, \tag{1.8}$$

$$A \cap B = B \cap A, \tag{1.9}$$

Distributiveness

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \tag{1.10}$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) \tag{1.11}$$

And for the sum and intersection of a **set** A with itself and with its **subset** B, the relations are valid

$$A \cup A = A, \tag{1.12}$$

$$A \cap A = A, \tag{1.13}$$

$$A \cup B = A, \tag{1.14}$$

$$A \cap B = B \tag{1.15}$$

The **set** Q consisting of the elements of the **set** A that are not included in the **set** B is called the difference of these **sets** and is denoted as $A - B$ or $A \setminus B$, i.e.

$$Q = A - B = A \setminus B \tag{1.16}$$

It is quite obvious that in the general case

$$A - B \neq B - A \tag{1.17}$$

For example, for numerical **sets** (1.2)

$$A - B = \{1,2,3\}, \tag{1.18}$$

$$B - A = \{6,7,8\} \tag{1.19}$$

If the **set** A is a **subset** of the **set** B, then the difference $B - A$ is called the complement of the **set** A to the **set** B and is symbolically denoted as $C_B A$, i.e.

$$C_B A = B - A \tag{1.20}$$

For example, the **set** $\overline{R}$ of irrational numbers on the number line is the complement of the **set** R of rational numbers to the **set** Z of real numbers, i.e.

$$\overline{R} = C_Z R = Z - R \tag{1.21}$$

If the **sets** A_i, $i = 1,\ 2,\ ...,\ n$ are **subsets** of the **set** A, then the relations are valid

$$C_A A_1 \cup C_A A_2 \cup ... \cup C_A A_n = C_A (A_1 \cap A_2 \cap ... \cap A_n), \tag{1.22}$$

$$C_A A_1 \cap C_A A_2 \cap ... \cap C_A A_n = C_A (A_1 \cup A_2 \cup ... \cup A_n), \tag{1.23}$$

the correctness of which is easy to verify graphically, for example, for $n=3$, if the set A is represented in the figure as a square with three circles inscribed in it, representing the subsets of A_1, A_2, A_3.

An important characteristic of **sets** is their **equivalence**, according to which **sets** *A,B* are considered as **equivalent** if, according to some rule, each element $a \in A$ is matched by one unique element $b \in B$, and each element $b \in B$ is matched by one unique element $a \in A$. For example, a **set** *A* of privately owned passenger cars, each of which is registered to only one owner in a certain settlement, and a **set** *B* of people who own these cars are equivalent. The rule by which the equivalence of these sets is established is the entry of the owner's surname in the vehicle passport.

And in order to compare **non-equivalent sets**, the concept of their **power** is introduced, which for the **set** A is symbolically written as $\overline{\overline{A}}$ and which is determined by something common that occurs in all **sets equivalent** to the one under consideration. It is obvious that **finite sets** of different natures have only the number of their elements in common, and therefore, if a **set** *A* has ***n*** of elements, and a **set** *B* has ***m*** of elements and at the same time $n > m$, then we state that the **set** *A* has a power greater than the **set** *B*.

But there arises a question: "And how to compare the **power of infinite sets**, each of which has an infinite number of elements?".

In mathematics, it is established that of all infinite sequences, the **natural series** *N* approaches infinity the fastest since each of its subsequent numbers is equal to the previous number increased by one, and at the same time, when forming this series, all real numbers that are contained on the number axis in each such unit are omitted. And therefore the **natural series**, which is an infinite series of numbers, is **an infinite set of the lowest power, symbolically denoted by a small Latin letter** a, that is,

$$\overline{\overline{N}} = a, \tag{1.24}$$

and all other **infinite sets** will be compared among themselves, based on how they are related by **power** to the **power of the natural series**, determined by the relation (1.24). And all **infinite sets** with the **power of a natural series** are called **countable sets**, since each of their elements can be assigned an index equal to the corresponding number of the natural series, due to which each of their elements can be **counted**.

And the first fact that was established in mathematics after the agreement regarding the **power of the natural series** is that **the power of the set *Z* of real number**s given on the interval [0,1] is **bigger than** a.

The proof of this fact is simple - if you add a sequence of real numbers $x_1, x_2, x_3, \ldots$ on the segment [0,1] of the numerical axis so that each subsequent number is three times smaller than the previous one, and divide this segment [0,1] into three equal segments Δ_1 each with a width of at least one a point x_1 will not enter from these segments. Let's divide the segment in which the point x_1 did not enter, also into three equal segments of width Δ_2 each, and choose the one from them in which the point x_2. did not enter. According to this algorithm, we will continue this process ad infinitum. As a result, on the segment [0,1] of the numerical axis, we will receive a **counted set** $\Delta_1, \Delta_2, \Delta_3, \ldots,$, the elements of which are smaller and smaller segments of the segment, and next to which, on the same segment, there is a previously calculated set of numbers $x_1, x_2, x_2, \ldots,$, none of which falls into any of these segments . And this means that there are more real numbers on the segment [0,1] of the number axis than there are numbers of the natural series on the entire number axis, which allows us to conclude that the **power of an infinite set of real numbers on the segment [0,1] is greater than the power of the natural series, which is a countable set.**

In mathematics, **the power of an infinite set of real numbers on the interval [0,1] is called the power of a continuum denoted by a lowercase Latin letter *c***, therefore, the inequality is valid

$$c > a. \tag{1.25}$$

Moreover, it was established that for powers *c, a* the inequality (1.25), as well as the equality

$$c = 2^a. \tag{1.26}$$

are real.

To prove the equality (1.26), we will use the method of mathematical induction according to the algorithm: we will consider successively which powers $n_0,\ n_1,\ n_2,\ n_3$ **will have sets generated by finite sets** $A_0 = \{O\}, A_1 = \{a_1\}, A_2 = \{a_1, a_2\}, A_3 = \{a_1, a_2, a_3\}$, if all **possible subsets** generated by the elements of these **finite sets** are introduced as elements into each of the generated sets. Bearing in mind that the number of combinations of elements of C from the *n* of the elements on *m, as* is known from the high school mathematics, shall be determined according to correlation

$$C_n^m = \frac{n!}{m!\,(n-m)!}, \tag{1.27}$$

we will find that:

$$\begin{cases} n_0 = C_0^0 = \dfrac{0!}{0!0!} = 1 = 2^0, \\ n_1 = C_1^0 + C_1^1 = \dfrac{1!}{0!1!} + \dfrac{1!}{1!0!} = 1 + 1 = 2 = 2^1, \\ n_2 = C_2^0 + C_2^1 + C_2^2 = \dfrac{2!}{0!2!} + \dfrac{2!}{1!1!} + \dfrac{2!}{2!0!} = 1 + 2 + 1 = 4 = 2^2, \\ n_3 = C_3^0 + C_3^1 + C_3^2 + C_3^3 = \dfrac{3!}{0!3!} + \dfrac{3!}{1!2!} + \dfrac{3!}{2!1!} + \dfrac{3!}{3!0!} = 1 + 3 + 3 + 1 = 8 = 2^3. \end{cases} \tag{1.28}$$

According to the **ideology of the method of mathematical induction**, it **follows from relations (1.28) that if a finite set** $A_n = \{a_1,\ a_2,\ a_3,\ ...,\ a_n\}$ **has *n* elements, then the power** n_n of **the set generated by it**, which includes **all possible subsets of this set, will be equal to**

$$n_n = 2^n. \tag{1.29}$$

And hence the conclusion that if the **power of the counted set is** a, **equal to the power of the infinite set of real numbers generated by it on the interval [0, 1], the elements of which are all possible subsets of the elements of this set, will be equal to two to the power of** a, which proves the validity of the equality (1.26).

And now let's return to the expression (1.21), according to which the **set *Z* of real numbers on the segment [0, 1]** of the numerical axis is the **sum of the subset *R* of rational numbers and the subset** $\overline{R}$ **of irrational numbers** given on the same segment.

As is known, each rational number is the ratio of two integers, and if this rational number is less than one, then its numerator is always an integer that is smaller than the number in the denominator. Since the integers are elements of the natural series, which is a countable power set, then these numbers can be counted both in the numerator and in the denominator, and **therefore the subset of rational numbers on the segment [0, 1] of the**

number axis is also a countable set of power a. The above fact has **two consequences**, **the first** of which proves that **the subset of irrational numbers on the specified segment is an infinite set of the power of the continuum** c, because only due to this subset the set of real numbers on the specified segment will have the power c that we have already shown above with respect to the set of real numbers. And **the second consequence is the statement that if any counted subset is added to the power set of the continuum, the power of their sum remains equal** c.

And then we pay attention to the fact that all unit segments on the number axis, located between adjacent natural numbers, can be counted by assigning to each of them an index equal to the natural number placed on the right border of each such unit segment, so a subset of unit segments, placed between natural numbers on the number axis, is a counted set of power a,, which is a smaller power of the continuum of the unit segment [0, 1] of the number axis. So, based on this statement, we can draw an important conclusion that **the entire axis of real numbers is a multiple of the power of the continuum** c**.**

But, as we have already shown above, the **set that is generated by the union of all possible subsets of the generic set of a certain power has a power equal to two to the power equal to the power of the generic set**. And from this fact we draw the conclusion: **the power** f **of the set of all functions** $f(x)$ **the argument** x **of which is set on the segment [0, 1]** (or on the entire numerical axis) **of the power of the continuum** c **is equal to two in the power of** c**, that is**,

$$\bar{\bar{f}} = 2^c \tag{1.30}$$

This is where we will finish the consideration of the material of the subsection dedicated to sets, subsets and their characteristics, which we will need when explaining the basics of functional analysis. Those who wish to learn more about this area of mathematics are referred to textbooks on set theory or functional analysis, which are used by students of mathematical specialties at universities.

1.2 Metric spaces and their classes and characteristics

Set

$$\Omega = \{x, y, z, \ldots, u, v, \ldots\} \tag{1.31}$$

of the elements of some nature are called **metric space** if each ordered pair of elements $x, y \in \Omega$ is in line with an integral number $\rho(x, y)$, which is called the **metric of space** Ω, if this number satisfies three axioms of metrics:

1) axiom identity

$$\rho(x, y) = 0 \tag{1.32}$$

then and only then, when

$$x = y; \tag{1.33}$$

2) axiom symmetry

$$\rho(x, y) = \rho(y, x); \tag{1.34}$$

3) the triangle axiom

$$\rho(x, y) + \rho(y, z) \geq \rho(x, z). \tag{1.35}$$

Considering these axioms we see that the **metric** $\rho(x, y)$ of **space** Ω **sets the distance between the elements** x **,** y **of this space.**

The elements of the metric space are called points.

Examples

1. For a three-dimensional Euclidean space E_3 the distance between points $x = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2, y_3\}$ ($x, y \in E_3$) is determined by an expression

$$\rho(x, y) = \sqrt{\sum_{i=1}^{3}(x_i - y_i)^2}. \tag{1.36}$$

2. For the set $C[0,1]$ of continuous functions $x(t)$, $y(t)$, ..., given on the segment $t \in [0,1]$, the distance between the elements $x(t)$ and $y(t)$ is given by the expression

$$\rho(x, y) = \max_t |x(t) - y(t)|. \tag{1.37}$$

If X – an arbitrary metric space then the sequence

$$\{x_n\} \subset X \tag{1.38}$$

coincides to a point $x_0 \in X$, if when $n \to \infty$

$$\rho(x_n, x_0) \to 0, \tag{1.39}$$

or, as written otherwise

$$\lim_{n \to \infty} x_n = x_0. \tag{1.40}$$

The sequence $\{x_n\}$, that coincides to some point x_0, is limited.

If the set contains all its limit points, then it is closed.

Let the metric space X is given, and let there be a sequence of points $\{x_n\}$ in this space that coincides to the point $x_0 \in X$. Then, when $n \to \infty$ the expression (1.39) will be fair as well as the expression

$$\rho(x_{n+p}, x_0) \to 0, \tag{1.41}$$

for and any $p > 0$. And the inequality of the triangle (1.35) using expressions (1.39) and (1.41) takes a form of

$$\rho(x_{n+p}, x_0) + \rho(x_n, x_0) \geq \rho(x_{n+p}, x_n). \tag{1.42}$$

And from expressions (1.39), (1.41) and (1.42) due to the inequality of a triangle for metrics, we will have an expression

$$\rho(x_{n+p}, x_n) \to 0. \tag{1.43}$$

If a condition (1.43) is fulfilled for some sequence $\{x_n\} \subset X$, then it is called a **fundamental sequence** or sequence that coincides in itself or a **sequence of Cauchy**.

If in the metric space ***X***, any sequence $\{x_n\} \subset X$,that coincides to itself, coincides to some limiting point x_0, which is an element of the same space, that is $x_0 \in X$, then this space ***X*** is called **complete**.

Metric space *X* is called linear if it defines the operations of addition and multiplication by the scalar, which satisfy the following conditions:

1) $$x^* + x^{**} = x^{**} + x^*,\ \forall\, x^*, x^{**} \in X; \qquad (1.44)$$

2) $$\left(x^* + x^{**}\right) + x^{***} = x^* + \left(x^{**} + x^{***}\right),\ \forall\, x^*, x^{**}, x^{***} \in X; \qquad (1.45)$$

3) $$x + 0 = x,\ \forall\, x \in X, 0 \in X, \qquad (1.46)$$

where the element **0** is zero of the set **X**;

4) for $\forall\, x^* \in X$ there is $x^{**} \in X$ such that

$$x^* + x^{**} = 0, \qquad (1.47)$$

where the element x^{**} is an element opposite to the element x^*;

5) $$1 \cdot x = x,\ \forall\, x \in X; \qquad (1.48)$$

6) $$\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x,\ \forall\, x \in X \text{ and } \forall\, \alpha, \beta; \qquad (1.49)$$

7) $$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x,\ \forall\, x \in X \text{ and } \forall\, \alpha, \beta; \qquad (1.50)$$

8) $$\alpha \cdot \left(x^* + x^{**}\right) = \alpha \cdot x^* + \alpha \cdot x^{**},\ \forall\, x^*, x^{**} \in X \text{ and } \forall\, \alpha. \qquad (1.51)$$

A linear metric space ***X*** is called normalized if $\forall\, x \in X$ it can be matched by some non-negative number $\| x \|$, which is called the norm and which satisfies the following conditions:

1) $$\| x \| = 0 \text{ if and only if } x = 0; \qquad (1.52)$$

2) $$\| \alpha \cdot x \| = | \alpha | \cdot \| x \|, \quad \alpha \text{ is a scalar;} \qquad (1.53)$$

3) $$\left\| x^* + x^{**} \right\| \le \left\| x^* \right\| + \left\| x^{**} \right\|,\ \forall\, x^*, x^{**} \in X. \qquad (1.54)$$

It is quite obvious that the norm $\| x \|$ is the distance from the element ***x*** to the zero element of the set ***X***.

Examples of norms:

1) for space $C[0,1]$

$$\| x(t) \| = \max_{t \in [0,1]} | x(t) | \qquad (1.55)$$

or

$$\| x(t) \| = \sup_{t \in [0,1]} | x(t) |; \qquad (1.56)$$

2) for the Euclidean dimension E_n of the space n

$$\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2}\ , \tag{1.57}$$

where $x = \{x_1,\ x_2,\ \dots,\ x_n\},\ x \in E_n$.

It is obvious that for any linear normalized space ***X*** the relation is valid

$$\left\| x^* - x^{**} \right\| = \rho\left(x^*, x^{**} \right), \tag{1.58}$$

where $x^*, x^{**} \in X$.

A complete linear normalized space is called Banach (after the name of the mathematician who studied this space) and is denoted as B-space.

It is clear that the spaces $C[0,1]$ and E_n are Banach.

Note that the norm in B-space can be introduced in different ways, so long as it meets the conditions (1.52), (1.53), (1.54).

For example, in the space of functions $x(t)$ continuous on a segment $t \in [0,1]$, the norm can be introduced not only in the form (1.55), but also in the form

$$\| x \| = \int_0^1 |x(t)|\, dt\ . \tag{1.59}$$

Such a B-space is called a Lebesgue space which is denoted by $L[0,1]$, to distinguish it from the space $C[0,1]$ of the same functions, but with norm (1.55).

For space E_n as a norm, you can use not only the ratio (1.72), but also a more general one

$$\| x \| = \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}},\ \ p > 0\ . \tag{1.60}$$

It is clear that (1.57) coincides to (1.60) for $p = 2$.

A Banach space with a scalar product of elements is called a Hilbert space (after the name of the mathematician who studied it) and is denoted as an H-space.

H-space can be finite-dimensional or infinite-dimensional.

The scalar product of the elements $f, g \in H$ is written in the form (f, g) or $\langle f, g \rangle$. The scalar product must be subject to the following conditions:

1) $$\langle f, g \rangle = \langle g, f \rangle, \tag{1.61}$$

2) $$\langle \alpha \cdot f, g \rangle = \alpha \cdot \langle f, g \rangle, \tag{1.62}$$

3) $$\langle f, \alpha \cdot g \rangle = \alpha \cdot \langle f, g \rangle, \tag{1.63}$$

4) $$\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle; \tag{1.64}$$

5) $$\langle f, g_1 + g_2 \rangle = \langle f, g_1 \rangle + \langle f, g_2 \rangle; \quad (1.65)$$

6) $$\langle f, f \rangle > 0 \text{, якщо } f \neq 0. \quad (1.66)$$

It follows from the expression for the norm that for the H-space

$$\| f \| = \sqrt{\langle f, f \rangle}. \quad (1.67)$$

H-space is often considered in two implementations.

1. Space l_2 of all counted ordered sequences $x \in l_2$

$$x = \{x_1, x_2, \ldots, x_n, \ldots\} \quad (1.68)$$

such that have the property

$$\sum_{i=1}^{\infty} x_i^2 < \infty. \quad (1.69)$$

For elements: $x^*, x^{**} \in l_2$:

$$\rho(x^*, x^{**}) = \sqrt{\sum_{i=1}^{\infty} \left(x_i^* - x_i^{**}\right)^2}; \quad (1.70)$$

$$\| x^* \| = \sqrt{\sum_{i=1}^{\infty} \left(x_i^*\right)^2}; \quad (1.71)$$

$$\| x^* - x^{**} \| = \rho(x^*, x^{**}); \quad (1.72)$$

$$\langle x^*, x^{**} \rangle = \sum_{i=1}^{\infty} x_i^* \cdot x_i^{**}; \quad (1.73)$$

$$\| x^* \| = \sqrt{\langle x^*, x^* \rangle} \quad (1.74)$$

It follows from these relations that l_2-space is a generalization of Euclidean E_n-space when $n \to \infty$.

l_2 **-space is sometimes called a coordinated Hilbert space.**

2. **The space** $L_2[a, b]$ **of functions** $f(t)$ **with an integrated square**, that is, for which

$$\int_a^b f^2(t)\, dt < \infty. \quad (1.75)$$

The following relations are valid for $f(t), g(t) \in L_2[a, b]$:

$$\rho(f, g) = \sqrt{\int_a^b \left(f(t) - g(t)\right)^2 dt}; \quad (1.76)$$

$$\| f\| = \sqrt{\int_a^b f^2(t)\, dt}\ ; \tag{1.77}$$

$$\| f - g\| = \rho(f, g); \tag{1.78}$$

$$\langle f, g\rangle = \int_a^b f(t)\cdot g(t)\, dt\ . \tag{1.79}$$

Let's write two widely used inequalities separately:

the Cauchy–Minkowski inequality

$$\| f + g\| \le \| f\| + \| g\|, \tag{1.80}$$

the Buniakovsky–Schwartz inequality

$$|\langle f, g\rangle| \le \| f\|\cdot\| g\|, \tag{1.81}$$

to prove which it is enough to substitute expressions for all components in them.

1.3 Orthonormal subsets in Hilbert spaces

Consider a functional Hilbert space $H[a,b]$ such that $x(t), y(t) \in H[a,b]$, $t \in [a,b]$.

Let the scalar product $\langle x, y\rangle$ of functions $x(t)$ and $y(t)$ equal to zero, that is,

$$\langle x, y\rangle = \int_a^b x(t)\cdot y(t)\, dt = 0\,. \tag{1.82}$$

If the condition (1.82) is satisfied for the functions $x(t), y(t) \in H[a,b]$, **then they are said to be orthogonal on** $[a,b]$.

Let us have in the Hilbert space $H[a,b]$ a finite-dimensional or infinite sequence of functions $\{\varphi_k(t)\}$ such that

.
$$\{\varphi_k(t)\} \subset H[a,b],\ t \in [a,b]. \tag{1.83}$$

If the condition is true for this sequence $\{\varphi_k(t)\}$

$$\langle \varphi_k, \varphi_m\rangle = \int_a^b \varphi_k(t)\cdot\varphi_m(t)\, dt = 0,\ k \ne m\,, \tag{1.84}$$

then this sequence is called orthogonal.

If the condition is satisfied for an orthogonal sequence $\{\varphi_k(t)\} \subset H[a,b]$

$$\int_a^b \varphi_k^2(t)\,dt = 1, \tag{1.85}$$

then this sequence is called orthonormal.

A sequence $\{\varphi_k(t)\} \subset H[a,b]$ is called orthogonal with weight $w(t)$, if there exists a function $w(t) \in H[a,b]$, that satisfies the condition

$$\int_a^b \varphi_k(t)\cdot\varphi_m(t)\cdot w(t)\,dt = 0,\ k \neq m. \tag{1.86}$$

It is clear that the sequence $\left\{\sqrt{w(t)}\cdot\varphi_k(t)\right\} \subset H[a,b]$ is simply orthogonal.

A subset of orthogonal functions $\{\varphi_k(t)\} \subset H[a,b]$ is complete in H-space if there is no nonzero function in it that would be orthogonal to any of the functions of this sequence.

A sequence of functions $\{\varphi_k(t)\} \subset H[a,b]$ is called closed in H-space if for $\forall\, f(t) \in H[a,b]$ and for $\forall\, \varepsilon > 0$ it is possible to construct such a linear combination of functions $\varphi_k(t)$, taken with weight λ_k, that the condition is fulfilled

$$\| f(t) - \lambda_1\cdot\varphi_1(t) - \lambda_2\cdot\varphi_2(t) - \cdots - \lambda_k\cdot\varphi_k(t) - \cdots \| < \varepsilon. \tag{1.87}$$

This means that with an error that does not exceed ε, the function $f(t) \in H[a,b]$ on the segment $[a,b]$ can be presented in the form

$$f(t) \cong \sum_{k=1}^{N} \lambda_k \cdot \varphi_k(t), \tag{1.88}$$

where ***N*** can be either a finite integer or infinity.

Different mathematics for basic functions

$$f_k(t) = t^k,\ k = \overline{0,n} \tag{1.89}$$

obtained various systems of orthonormal polynomials for different weight functions and orthogonalization intervals. Therefore, it is not necessary to build this sequence yourself every time you need to approximate a function $f(t) \in H[a,b]$ using an orthonormal sequence $\{\varphi_k(t)\} \subset H[a,b]$. It is enough to choose one of those built by others, using a reference book on higher mathematics or a manual on the mathematical theory of processing the results of experiments.

Here are examples of orthogonalization intervals, weighting functions, and normalization factors of the most common systems of orthogonal polynomials (Table 1).

Table 1 - Examples of orthogonalization intervals, weighting functions, and normalization factors for the most common systems of orthogonal polynomials

orthogonal polynomials	orthogonalization intervals	weighting functions $W(t)$	normalization factors
Legendre $P_k(t)$	$t\in[-1,1]$	1	$\int_{-1}^{1}(P_k(t))^2\,dt=\frac{2}{2k+1}$
Chebyshov I $T_k(t)$	$t\in[-1,1]$	$(1-t^2)^{-\frac{1}{2}}$	$\int_{-1}^{1}(T_k(t))^2(1-t^2)^{-\frac{1}{2}}\,dt=\begin{cases}\frac{\pi}{2}, k\neq 0\\ \pi, k=0\end{cases}$
Chebyshov II $U_k(t)$	$t\in[-1,1]$	$(1-t^2)^{\frac{1}{2}}$	$\int_{-1}^{1}(U_k(t))^2(1-t^2)^{\frac{1}{2}}\,dt=\frac{\pi}{2}$
Laguerra $L_k(t)$	$t\in[0,\infty)$	e^{-t}	$\int_{0}^{\infty}(L_k(t))^2 e^{-t}dt=1$
Laguerra attached $L_k^{(i)}(t)$	$t\in[0,\infty)$	$t^i\cdot e^{-t}$	$\int_{0}^{\infty}\left(L_k^{(i)}(t)\right)^2 t^i e^{-t}dt=\frac{(k+1)!}{k!}$
Ermita $H_k(t)$	$t\in(-\infty,\infty)$	e^{-t^2}	$\int_{-\infty}^{\infty}(H_k(t))^2 e^{-t^2}dt=2^k\cdot k!\cdot\sqrt{\pi}$

Thus, in order to approximate the function $f(t)\in H[a,b]$, $t\in[a,b]$ using an orthonormal system of polynomials $\{\varphi_k(t)\}\subset H[a,b]$, it is necessary, based on the interval of orthogonalization $[a,b]$ and the convenience of the weight function $w(t)$, to select one or another orthonormal system of polynomials from the directory and find the ratio for the general member of the selected system, revealing which one to obtain the number of its members, which is sufficient to ensure the given accuracy of the approximation.

For example, we give an expression for a common member –

$$P_n(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \qquad n=0,1,2,\dots,N \qquad (1.90)$$

and the first 7 members of the orthonormal sequence for Legendre polynomials, the weighting function for which is the function $w(t)=1$, the orthogonalization interval is the segment [-1,1), the normalization factor has the form $2/(2n+1)$. Therefore, according to expression (1.90), we will have:

$$
\begin{Bmatrix}
P_0(t) = 1, \\
P_1(t) = t, \\
P_2(t) = \frac{1}{2}(3t^2 - 1), \\
P_3(t) = \frac{1}{2}(5t^3 - 3t), \\
P_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3), \\
P_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t), \\
P_6(t) = \frac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5), \\
P_7(t) = \frac{1}{16}(429t^7 - 693t^5 + 315t^3 - 35t)
\end{Bmatrix} \tag{1.91}
$$

As a second example, we give the formula for the general member –

$$T_n(t) = \frac{1}{2^n}\left(\left(t + \sqrt{t^2 - 1}\right)^n + \left(t - \sqrt{t^2 - 1}\right)^n\right), \qquad n = 1,2,\dots,N \tag{1.92}$$

and the first 7 members of the orthonormal sequence for Chebyshov 1 polynomials, the weighting function for which is the function $w(t) = \sqrt{1 - t^2}$, the orthogonalization interval is the segment [-1,1), the normalization factor has the form π for $k = 0$ and $\pi/2$ for $k \neq 0$. Therefore, according to expression (1.92), we will have:

$$
\begin{Bmatrix}
T_0(t) = 1, \\
T_1(t) = t, \\
T_2(t) = \frac{1}{2}(2t^2 - 1), \\
T_3(t) = \frac{1}{4}(4t^3 - 3t), \\
T_4(t) = \frac{1}{8}(8t^4 - 8t^2 + 1), \\
T_5(t) = \frac{1}{16}(16t^5 - 20t^3 + 5t), \\
T_6(t) = \frac{1}{32}(32t^6 - 48t^4 + 18t^2 - 1), \\
T_7(t) = \frac{1}{64}(64t^7 - 112t^5 + 56t^3 - 7t)
\end{Bmatrix} \tag{1.93}
$$

1.4 Approximation of continuous functions in Hilbert spaces

The approximation of continuous functions is understood as the process of finding an analytical description of a function given by the elements of some set, which may not be a subset of the selected space, in the selected space. For example, a polynomial approximation of a function given in the form of a table.

Let $\{\varphi_k(t)\} \subset H[a,b]$, $t \in [a,b]$, be some complete sequence of orthonormal functions that is closed in this space.

Let $H[a,b] = L[a,b]$ is the H-space of functions $f(t) \in L[a,b]$ for which the condition is satisfied

$$\int_a^b |f(t)|\, dt < \infty, \tag{1.94}$$

and the metric $\rho(f_1, f_2)$ is given by the ratio

$$\rho(f_1, f_2) = \int_a^b \left| f_1(t) - f_2(t) \right| dt. \tag{1.95}$$

Suppose that the series

$$\sum_k \lambda_k \cdot \varphi_k(t), \tag{1.96}$$

where λ_k is some scalar unknown to us, which converges uniformly to some function $f(t) \in L[a, b]$. This means that for $\forall\, \varepsilon > 0$ exists such that ***m*** for $\forall\, t \in [a, b]$ and $\forall\, n \geq m$ the relation holds

$$\int_a^b \left| f(t) - \sum_{k=0}^{n} \lambda_k \cdot \varphi_k(t) \right| dt < \varepsilon, \tag{1.97}$$

from which it $n \to \infty$ follows that

$$f(t) = \sum_{k=0}^{\infty} \lambda_k \cdot \varphi_k(t). \tag{1.98}$$

To determine the weighting coefficients $\lambda_k,\ k = \overline{0, \infty}$, multiply both parts of equation (2.41) by $\varphi_j(t)$ and integrate the result in the range from «a» to «b».. As a result, we get:

$$\int_a^b f(t) \cdot \varphi_j(t)\, dt = \sum_{k=0}^{\infty} \lambda_k \cdot \int_a^b \varphi_k(t) \cdot \varphi_j(t)\, dt. \tag{1.99}$$

Since $\{\varphi_k(t)\} \subset L[a, b]$ it is an orthonormal sequence, relations (1.84) and (1.85) hold for it. Taking this into account, from (1.99) we have

$$\lambda_j = \int_a^b f(t) \cdot \varphi_j(t)\, dt, \ \ j = 0, 1, 2, \ldots. \tag{1.100}$$

The weighting coefficients λ_j are called Fourier coefficients, and their complete sequence $\{\lambda_j\}$ is called the Fourier spectrum of the expansion of a function $f(t) \in L[a, b]$ by an orthonormal system of functions $\{\varphi_j(t)\} \subset L[a, b]$.

The requirement (1.97) of uniform convergence of the series (1.96) to the function $f(t)$ is the so-called "strong convergence requirement".

But it turns out that in H-space the strong convergence is equivalent to "convergence on the average", which is a weaker requirement and can be written as

$$\lim_{n \to \infty} \int_a^b \left[f(t) - \sum_{k=0}^{n} \lambda_k \cdot \varphi_k(t) \right]^2 dt = 0. \tag{1.101}$$

We consider the process of approximating a function $f(t) \in L_2[a, b]$ in H-space $L_2[a, b]$ using an orthonormal sequence $\{\varphi_k(t)\} \subset L_2[a, b]$.

In this case, the approximation problem can be reduced to such a selection of partial sum coefficients C_k

$$S_n(t)=\sum_{k=0}^{n} C_k \cdot \varphi_k(t) \tag{1.102}$$

in the H-space $L_2[a,b]$ so that this sum approaches the function $f(t)\in L_2[a,b]$ with an error not exceeding the given one, i.e. so that

$$\|f(t)-S_n(t)\|=\sqrt{\int_a^b [f(t)-S_n(t)]^2\,dt} \to \min_{C_k}. \tag{1.103}$$

To find $\min\limits_{C_k}$ of the expression (1.103), we compose and solve the system of equations

$$\frac{\partial E}{\partial C_k}=0,\ k=\overline{0,n}, \tag{1.104}$$

where

$$\begin{gathered} E=\int_a^b [f(t)-S_n(t)]^2\,dt= \\ =\int_a^b f^2(t)\,dt-2\cdot\int_a^b f(t)\cdot S_n(t)\,dt+\int_a^b [S_n(t)]^2\,dt. \end{gathered} \tag{1.105}$$

As a result of solving the system of equations (1.104), we find that

$$C_k=\lambda_k. \tag{1.106}$$

Therefore, in order for the partial sum $S_n(t)$ to approximate the function $f(t)$ with the specified accuracy, it is necessary to choose the Fourier coefficients λ_k of the function $f(t)$ as coefficients C_k.

Substituting (1.106) into (1.105), we will have:

$$E=\int_a^b f^2(t)\,dt-\sum_{k=0}^{n}\lambda_k^2 \ge 0. \tag{1.107}$$

Because

$$\lim_{n\to\infty} E=0, \tag{1.108}$$

then it follows from the expression (1.107) that

$$\int_a^b f^2(t)\,dt=\sum_{k=0}^{\infty}\lambda_k^2. \tag{1.109}$$

The relation (1.109) is called Parseval's equality. The square root of both its parts can be interpreted as the length of the vector $f(t)$ in the *H*-space $L_2[a,b]$, expressed through its projections on the orthogonal coordinate system $\{\varphi_k(t)\}$, which is a subset of the same H-space $L_2[a,b]$.

Concluding this subsection, we emphasize that in case of using the Legendre orthonormal polynomial function (1.91) for approximation, the Fourier coefficients must be calculated not by the expression (1.100), but by the expression

$$\mu_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)dt, \quad n = 0,1,2,\dots,N, \qquad (1.110)$$

and in the case of using Chebyshev 1 (1.93) to approximate the function $f(t)$ of orthonormal polynomials, the Fourier coefficients must be calculated not by the expression (1.100), but by the expressions:

$$\mu_n = \frac{1}{\pi}\int_{-1}^{1} f(t)T_n(t)(1-t^2)^{-\frac{1}{2}}\,dt, \quad n = 0, \qquad (1.111)$$

$$\mu_n = \frac{2}{\pi}\int_{-1}^{1} f(t)T_n(t)(1-t^2)^{-\frac{1}{2}}\,dt, \quad n = 1,2,\dots,N \qquad (1.112)$$

1.5 Programs for implementing operations in metric spaces in Python

A Python program for checking sets for equality, determining their power, and checking for equivalence

(Program 1):

```
In [1]: A={1,2,3,4,5}
In [2]: B={4,5,6,7,8}
In [3]: LA=list(A); LA
Out[3]: [1, 2, 3, 4, 5]
In [4]: LB=list(B); LB
Out[4]: [4, 5, 6, 7, 8]
In [5]: LA==LB
Out[5]: False
In [6]: len(LA)
Out[6]: 5
In [7]: len(LB)
Out[7]: 5
In [8]: len(LA)==len(LB)
Out[8]: True
```

End of program 1

A Python program for finding the sum of sets and their union excluding common elements, as well as for determining the difference and intersection of sets

(Program 2):

```
In [1]: dLA = {}
In [2]: dLA['a']=1
In [3]: dLA['b']=2
In [4]: dLA['c']=3
In [5]: dLA['e']=4
In [6]: dLA['h']=5
In [7]: dLA
Out[7]: {'a': 1, 'b': 2, 'c': 3, 'e': 4, 'h': 5}
In [8]: dLB={}
In [9]: dLB['e']=4;dLB['h']=5;dLB['p']=6;\
        dLB['q']=7; dLB['r']=8
In [10]: dLB
Out[10]: {'e': 4, 'h': 5, 'p': 6, 'q': 7, 'r': 8}
In [11]: dLA.keys() | dLB.keys()
Out[11]: {'a', 'b', 'c', 'e', 'h', 'p', 'q', 'r'}
In [12]: dLA.keys() - dLB.keys()
Out[12]: {'a', 'b', 'c'}
In [13]: dLB.keys() - dLA.keys()
Out[13]: {'p', 'q', 'r'}
In [14]: dLA.keys() & dLB.keys()
Out[14]: {'e', 'h'}
In [15]: dLA.keys() ^ dLB.keys()
Out[15]: {'a', 'b', 'c', 'p', 'q', 'r'}
```

End of program 2.

A Python program for determining the norm and metric of Banach spaces whose elements are numbers

(Program 3):

```
In [1]: import numpy as np
In [2]: a2=np.array([1,2])
In [3]: a3=np.array([1,2,3])
In [4]: a4=np.array([1,2,3,4])
In [5]: c2=np.array([2,1])
In [6]: c3=np.array([3,2,1])
In [7]: c4=np.array([4,3,2,1])
In [8]: e2=a2-c2;e2
Out[8]: array([-1,  1])
In [9]: e3=a3-c3;e3
Out[9]: array([-2,  0,  2])
In [10]: e4=a4-c4;e4
Out[10]: array([-3, -1,  1,  3])
In [11]: import scipy
In [12]: import scipy.linalg as la
In [13]: la.norm(a2)
Out[13]: 2.23606797749979
In [14]: la.norm(a3)
Out[14]: 3.7416573867739413
In [15]: la.norm(a4)
Out[15]: 5.477225575051661
In [16]: la.norm(c2)
Out[16]: 2.23606797749979
In [17]: la.norm(c3)
Out[17]: 3.7416573867739413
In [18]: la.norm(c4)
Out[18]: 5.477225575051661
In [19]: m2=la.norm(e2);m2
Out[19]: 1.4142135623730951
In [20]: m3=la.norm(e3);m3
Out[20]: 2.8284271247461903
In [21]: m4=la.norm(e4);m4
Out[21]: 4.47213595499958
```

End of program 3.

A Python program for determining the norms and metrics of Banach spaces C[0,1] whose elements are functions
(Program 4):

```
In [1]: import numpy as np
In [2]: x=np.linspace(0,1,11)
In [3]: g1=lambda x: -1+3*x-x**2
In [4]: g1vec=np.vectorize(g1)
In [5]: g11=g1vec(x)
In [6]: g11
Out[6]: array([-1.  , -0.71, -0.44, -0.19, 0.04,
              0.25, 0.44,  0.61,  0.76,  0.89,  1.])
In [7]: g111=np.piecewise(g11,[g11<0,g11>=0],\
       [lambda g11:-g11,lambda g11: g11])
In [8]: g111
Out[8]: array([1.  , 0.71, 0.44, 0.19, 0.04, 0.25,
              0.44, 0.61, 0.76, 0.89, 1.  ])
In [9]: ng1=g111.max( );ng1
Out[9]: 1.0
In [10]: ig1=g111.argmax( );ig1
Out[10]: 0
In [11]: g2=lambda x: 5*x-6*x**2
In [12]: g2vec=np.vectorize(g2)
In [13]: g22=g2vec(x);g22
Out[13]: array([ 0.   , 0.44, 0.76, 0.96,1.04,1.   ,
               0.84,  0.56,  0.16,  -0.36, -1.   ])
In [14]: g222=np.piecewise(g22,[g22<0,\
         g22>=0], [lambda g22:-g22,\
         lambda g22: g22])
In [15]: g222
Out[15]: array([0.   , 0.44, 0.76, 0.96, 1.04,1.   ,
                0.84, 0.56, 0.16, 0.36, 1.   ])
In [16]: ng2=g222.max( ); ng2
Out[16]: 1.0399999999999998
In [17]: ig2=g222.argmax( ); ig2
Out[17]: 4
In [18]: g3=lambda x: -1-2*x+5*x**2
In [19]: g3vec=np.vectorize(g3)
In [20]: g33=g3vec(x);g33
Out[20]: array([-1.   , -1.15, -1.2  , -1.15, -1.   ,
                -0.75, -0.4  , 0.05, 0.6 , 1.25, 2. ])
In [21]: g333=np.piecewise(g33,[g33<0,\
         g33>=0], [lambda g33:-g33,\
         lambda g33: g33])
In [22]: g333
Out[22]: array([1.  , 1.15, 1.2 , 1.15, 1.  , 0.75,
                0.4 , 0.05, 0.6 , 1.25, 2.  ])
In [23]: mg3=g333.max( );mg3
Out[23]: 2.0
In [24]: ig3=g333.argmax( );ig3
Out[24]: 10
```

End of program 4 .

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